METAMORPHOSIS FROM THE VIEWPOINT OF DIFFERENTIAL GEOMETRY

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We give the prescription for generating all transformations of a certain type (conformal) relating two-dimensional dual potentials and indicate mechanisms for nonuniqueness of the duality.

Coupling constant metamorphosis (CCM) has been the subject of several recent papers [1–4] dealing with the problem of integrability of Hamiltonian systems. The CCM usually involves some time-independent canonical transformation(s) on a conservative Hamiltonian system and the *non-canonical* exchange of the roles of the energy and the coupling constant. All this is done in a way that leaves the kinetic energy form-invariant. The resulting potential is "dual" to the original one in the sense that the corresponding solutions and conserved quantities transform into each other. Clearly if one system is integrable, so is the other. Our earlier work dealt with the one-dimensional and spherically symmetric cases [5]. Cabral and Gallas [3] emphasized the nonuniqueness of the duality using the example of the Fokas–Lagerstrom potential for which a dual partner had already been discovered by Hietarinta et al. [1] Cabral and Gallas found another dual to the Fokas–Lagerstrom potential. A related important paper by Gazeau [6] dealing with duality in quantum mechanics has recently been brought to our attention.

In this Letter we briefly examine the two-dimensional case making use of the geometrization of Hamiltonian systems via the Jacobi metric [7,8]. We distinguish two different types of transformations for obtaining dual potentials, and hence clarify the origin of the nonuniqueness pointed out by Cabral and Gallas. One type involves canonical transformations which exchange the roles of coordinates and momenta; this type was used by Hiertarinta et al. [1]. Cabral and Gallas in their quantum mechanical version accomplished the same result by Fourier-transforming the Schrödinger equation. The transformations used by Hietarinta et al. [1] and Cabral and Gallas seem to involve considerable ingenuity in addition to luck and appear to be difficult to systematize. The other type involves conformal coordinate transformations and is rather straightforward in principle. We shall be concerned with this latter type only here, and we shall give the prescription for generating all such transformations. An example of this type is the Levi-Civita transformation discussed by Yoshida [4]. Hietarinta [2] suggested these transformations as a possible way to CCM but did not exploit them in this connection. We believe that our use of the Jacobi metric makes the principle involved conceptually very clear.

We consider the Hamiltonian

$$H=\tfrac{1}{2}(p_u^2+p_v^2)+\alpha V(u,v), \tag{1}$$

where α is the coupling constant, and we assume that $V(u,v)>0$. The Jacobi metric corresponding to (1) is

$$ds^2=[E-\alpha V(u,v)](du^2+dv^2) \tag{2}$$

$$=[E/V(u,v)-\alpha]V(u,v)(du^2+dv^2). \tag{3}$$

It is clear from (3) that if we can find a transformation $x=x(u,v)$ and $y=y(u,v)$ such that

$$V(u,v)(du^2+dv^2)=dx^2+dy^2, \tag{4}$$

then, after relabeling $\alpha\to-\hat{E}$, and $E\to-\hat{\alpha}$, we would obtain the dual potential

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$$\hat{V}(x,y)=1/V(u(x,y),v(x,y)). \tag{5}$$

From the point of view of differential geometry, transformation (4) is a transformation relating two sets of *isothermal* coordinates [9]. It is well-known that all such transformations are generated by holomorphic functions as follows: Let $z=x+iy=f(w)$ be a holomorphic function of w, and $w=u+iv=g(z)$ be its inverse, i.e.,

$$f(g(z))=z, \quad \text{and} \quad (\mathrm{d}f/\mathrm{d}w)(\mathrm{d}g/\mathrm{d}z)=1, \tag{6}$$

if w and z are corresponding points and the derivatives do not vanish. (For details see the inverse function theorem [10].) Then $x=\mathrm{Re}f(u,v)$, $Y=\mathrm{Im}f(u,v)$ takes us from the isothermal (u,v) system to the isothermal (x,y) system. The functions x and y are *harmonic conjugate*, and it is easy to see using the Cauchy–Riemann relations for x and y, that (4) will also be satisfied provided that

$$|\mathrm{grad}\,x|^2=|\mathrm{d}f/\mathrm{d}w|^2=V(u,v), \tag{7}$$

furthermore

$$|\mathrm{d}g/\mathrm{d}z|^2=\hat{V}(x,y). \tag{8}$$

Eq. (7) has the form of the Hamilton–Jacobi equation for a conservative system in cartesian coordinates, or the eikonal equation of geometrical optics. It is not an easy equation to solve in general and in our case we must also require that x be harmonic i.e.,

$$\triangle x=0. \tag{9}$$

A mathematically analogous problem is that of finding the potential function just outside a charged conductor of given surface charge distribution σ (where σ^2 is essentially our V). The function $y(u,v)$ must satisfy equations identical to (7) and (9) with x replaced by y.

It is instructive and far simpler to choose a suitable holomorphic function $f(w)$, use (7) to find the corresponding $V(u,v)$, then (8) to obtain the dual potential $\hat{V}(x,y)$. We shall discuss a few examples below.

(i) Let

$$z=w^{s+1}, \tag{10}$$

$s\neq-1$ and real, then from (7) we have that

$$V=(s+1)^2\rho^{2s}, \tag{11}$$

where $\rho=|w|$, while from (8) we obtain

$$\hat{V}=[(s+1)^2r^{2s/(s+1)}]^{-1}, \tag{12}$$

where $r=|z|$. We see that transformations generated by (10) relate radial power potentials. Furthermore if we let $q=2s$, $p=-2s/(s+1)$, we find that the powers q and p of the dual potentials satisfy the equation

$$q+p+qp/2=0, \tag{13}$$

as emphasized by Cabral and Gallas and Gazeau.

(ii) Eq. (13) breaks down if q or p is equal to -2 $(s=-1)$, however if we let

$$z=\log w, \tag{14}$$

we find that

$$V=1/\rho^2 \tag{15}$$

is dual to the degenerate potential

$$\hat{V}=\mathrm{e}^{2x}. \tag{16}$$

(iii) The generating functions may introduce extra parameters in the potential. As our last example we consider the Möbius transformation

$$z=\frac{aw+b}{cw+d}, \quad w=\frac{-dz+b}{cz-a}, \tag{17}$$

where $ad-bc\neq0$, and for simplicity a, b, c, d are real. Since $z(w)$ has the same form as $w(z)$, it is no surprise that V has the same form as $\hat{V}$,

$$V=\frac{(ad-bc)^2}{[(cu+d)^2+c^2v^2]^2},$$

$$\hat{V}=\frac{(ad-bc)^2}{[(cx-a)^2+c^2y^2]^2}. \tag{18}$$

If we now let $a=0$, we see that

$$\hat{V}=\frac{(b/c)^2}{r^4}. \tag{19}$$

Note that eq. (13), for $q=-4$, gives us $p=-4$, thus the r^{-4} potential is self-dual under (10), while under (17) it is dual to the non-central potential V given by (18) with $a=0$.

Naturally if the potential energy U is of the form

$$U(u,v)=\sum_{k=1}^{N}\alpha_k V_k(u,v), \tag{20}$$

following the steps in eqs. (2)–(5) with any one of the V_k's, say V_j, we would obtain, after letting $\alpha_j \to -\hat{E}$, and $E \to -\hat{\alpha}_j$, the dual potential

$$\hat{U}_j(x,y) = \sum_{k \neq j}^{N} \alpha_k V_k / V_j + \hat{\alpha}_j / V_j , \qquad (21)$$

thus now we have an equivalence class of $N+1$ potentials.

Summarizing then, we have given the prescription for generating all conformal-type transformations connecting two-dimensional dual potentials; and have indicated additional mechanisms for non-uniqueness of the duality (example (iii), and eqs. (20), (21)) to that obtained by Cabral and Gallas.

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